BENDING FREQUENCIES OF BEAMS, RODS, AND PIPES Revision K

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Introduction

The fundamental frequencies for typical beam configurations are given in Table 1. Higher frequencies are given for selected configurations.

Table 1. Bending Frequencies	
Configuration	Frequency (Hz)
Cantilever	$f_1 = \frac{1}{2\pi}\left[\frac{3.5156}{L^2}\right]\sqrt{\frac{EI}{\rho}}$ $f_2 = 6.268\, f_1$ $f_3 = 17.456\, f_1$
Cantilever with End Mass m	$f_1 = \frac{1}{2\pi}\sqrt{\frac{3EI}{(0.2235\,\rho\, L + m)L^3}}$
Simply-Supported at both Ends (Pinned-Pinned)	$f_n = \left[\frac{1}{2\pi}\right]\left[\frac{n\pi}{L}\right]^2\sqrt{\frac{EI}{\rho}}$, n = 1, 2, 3, ...
Free-Free	$f_1 = \frac{1}{2\pi}\left[\frac{22.373}{L^2}\right]\sqrt{\frac{EI}{\rho}}$ $f_2 = 2.757\, f_1$ $f_3 = 5.404\, f_1$
Fixed-Fixed	Same as Free-Free
Fixed - Pinned	$f_1 = \frac{1}{2\pi}\left[\frac{15.418}{L^2}\right]\sqrt{\frac{EI}{\rho}}$

where

E	is the modulus of elasticity.
I	is the area moment of inertia.
L	is the length.
ρ	is the mass density (mass/length).

The derivations and examples are given in the appendices per Table 2.

Table 2. Table of Contents

Appendix	Title	Mass	Solution
A	Cantilever Beam I	End mass. Beam mass is negligible	Approximate
B	Cantilever Beam II	Beam mass only.	Approximate
C	Cantilever Beam III	Both beam mass and the end mass are significant	Approximate
D	Cantilever Beam IV	Beam mass only.	Eigenvalue
E	Beam Simply-Supported at Both Ends I	Center mass. Beam mass is negligible.	Approximate
F	Beam Simply-Supported at Both Ends II	Beam mass only	Eigenvalue
G	Free-Free Beam	Beam mass only	Eigenvalue
H	Steel Pipe example, Simply Supported and Fixed-Fixed Cases	Beam mass only	Approximate
I	Rocket Vehicle Example, Free-free Beam	Beam mass only	Approximate
J	Fixed-Fixed Beam	Beam mass only	Eigenvalue

Reference

1. T. Irvine, Application of the Newton-Raphson Method to Vibration Problems, Vibrationdata Publications, 1999.

APPENDIX G

Free-Free Beam

Consider a uniform beam with free-free boundary conditions.

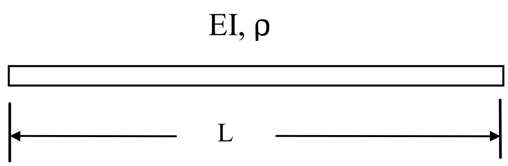

Figure G-1.

The governing differential equation is

$$-EI\frac{\partial^4 y}{\partial x^4} = \rho\frac{\partial^2 y}{\partial t^2} \tag{G-1}$$

Note that this equation neglects shear deformation and rotary inertia.

The following equation is obtain using the method in Appendix D

$$\frac{d^4}{dx^4}Y(x) - c^2\left\{\frac{\rho}{EI}\right\}Y(x) = 0 \tag{G-2}$$

The proposed solution is

$$Y(x) = a_1 \sinh(\beta x) + a_2 \cosh(\beta x) + a_3 \sin(\beta x) + a_4 \cos(\beta x) \tag{G-3}$$

$$\frac{dY(x)}{dx} = a_1\beta \cosh(\beta x) + a_2\beta \sinh(\beta x) + a_3\beta \cos(\beta x) - a_4\beta \sin(\beta x) \tag{G-4}$$

$$\frac{d^2 Y(x)}{dx^2} = a_1\beta^2 \sinh(\beta x) + a_2\beta^2 \cosh(\beta x) - a_3\beta^2 \sin(\beta x) - a_4\beta^2 \cos(\beta x) \tag{G-5}$$

$$\frac{d^3 Y(x)}{dx^3} = a_1\beta^3 \cosh(\beta x) + a_2\beta^3 \sinh(\beta x) - a_3\beta^3 \cos(\beta x) + a_4\beta^3 \sin(\beta x) \tag{G-6}$$

Apply the boundary conditions.

$$\left.\frac{d^2Y}{dx^2}\right|_{x=0} = 0 \qquad \text{(zero bending moment)} \tag{G-7}$$

$$a_2 - a_4 = 0 \tag{G-8}$$

$$a_4 = a_2 \tag{G-9}$$

$$\left.\frac{d^3Y}{dx^3}\right|_{x=0} = 0 \qquad \text{(zero shear force)} \tag{G-10}$$

$$a_1 - a_3 = 0 \tag{G-11}$$

$$a_3 = a_1 \tag{G-12}$$

$$\frac{d^2Y(x)}{dx^2} = a_1\beta^2\left[\sinh(\beta x) - \sin(\beta x)\right] + a_2\beta^2\left[\cosh(\beta x) - \cos(\beta x)\right] \tag{G-13}$$

$$\frac{d^3Y(x)}{dx^3} = a_1\beta^3\left[\cosh(\beta x) - \cos(\beta x)\right] + a_2\beta^3\left[\sinh(\beta x) + \sin(\beta x)\right] \tag{G-14}$$

$$\left.\frac{d^2Y}{dx^2}\right|_{x=L} = 0 \qquad \text{(zero bending moment)} \tag{G-15}$$

$$a_1[\sinh(\beta L) - \sin(\beta L)] + a_2[\cosh(\beta L) - \cos(\beta L)] = 0 \qquad \text{(G-16)}$$

$$\left.\frac{d^3Y}{dx^3}\right|_{x=L} = 0 \qquad \text{(zero shear force)} \qquad \text{(G-17)}$$

$$a_1[\cosh(\beta L) - \cos(\beta L)] + a_2[\sinh(\beta L) + \sin(\beta L)] = 0 \qquad \text{(G-18)}$$

Equation (G-16) and (G-18) can be arranged in matrix form.

$$\begin{bmatrix} \sinh(\beta L) - \sin(\beta L) & \cosh(\beta L) - \cos(\beta L) \\ \cosh(\beta L) - \cos(\beta L) & \sinh(\beta L) + \sin(\beta L) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{(G-19)}$$

Set the determinant equal to zero.

$$[\sinh(\beta L) - \sin(\beta L)][\sinh(\beta L) + \sin(\beta L)] - [\cosh(\beta L) - \cos(\beta L)]^2 = 0 \qquad \text{(G-20)}$$

$$\sinh^2(\beta L) - \sin^2(\beta L) - \cosh^2(\beta L) + 2\cosh(\beta L)\cos(\beta L) - \cos^2(\beta L) = 0 \qquad \text{(G-21)}$$

$$+2\cosh(\beta L)\cos(\beta L) - 2 = 0 \qquad \text{(G-22)}$$

$$\cosh(\beta L)\cos(\beta L) - 1 = 0 \qquad \text{(G-23)}$$

The roots can be found via the Newton-Raphson method, Reference 1. The first root is

$$\beta L = 4.73004 \qquad \text{(G-24)}$$

$$\omega_n = \beta_n^2 \sqrt{\frac{EI}{\rho}} \tag{G-25}$$

$$\omega_1 = \left[\frac{4.73004}{L}\right]^2 \sqrt{\frac{EI}{\rho}} \tag{G-26}$$

$$\omega_1 = \left[\frac{22.373}{L^2}\right] \sqrt{\frac{EI}{\rho}} \tag{G-27}$$

The second root is

$$\beta L = 7.85320 \tag{G-28}$$

$$\omega_n = \beta_n^2 \sqrt{\frac{EI}{\rho}} \tag{G-29}$$

$$\omega_2 = \left[\frac{7.85320}{L}\right]^2 \sqrt{\frac{EI}{\rho}} \tag{G-30}$$

$$\omega_2 = \left[\frac{61.673}{L^2}\right] \sqrt{\frac{EI}{\rho}} \tag{G-31}$$

$$\omega_2 = 2.757\ \omega_1 \tag{G-32}$$

The third root is

$$\beta L = 10.9956 \tag{G-33}$$

$$\omega_n = \beta_n^2 \sqrt{\frac{EI}{\rho}} \tag{G-34}$$

$$\omega_3 = \left[\frac{10.9956}{L}\right]^2 \sqrt{\frac{EI}{\rho}} \tag{G-35}$$

$$\omega_3 = \left[\frac{120.903}{L^2}\right]\sqrt{\frac{EI}{\rho}} \quad \text{(G-36)}$$

$$\omega_3 = 5.404\ \omega_1 \quad \text{(G-37)}$$

Equation (G-18) can be expressed as

$$a_2 = a_1\left[\frac{-\cosh(\beta L) + \cos(\beta L)}{\sinh(\beta L) + \sin(\beta L)}\right] \quad \text{(G-38)}$$

Recall

$$a_4 = a_2 \quad \text{(G-39)}$$

$$a_3 = a_1 \quad \text{(G-40)}$$

The displacement mode shape is thus

$$Y(x) = a_1[\sinh(\beta x) + \sin(\beta x)] + a_2[\cosh(\beta x) + \cos(\beta x)] \quad \text{(G-41)}$$

$$Y(x) = a_1\left\{[\sinh(\beta x) + \sin(\beta x)] + \left[\frac{-\cosh(\beta L) + \cos(\beta L)}{\sinh(\beta L) + \sin(\beta L)}\right][\cosh(\beta x) + \cos(\beta x)]\right\} \quad \text{(G-42)}$$

An alternate form is

$$Y(x) =$$

$$\hat{a}_1 \{[\sinh(\beta L) + \sin(\beta L)][\sinh(\beta x) + \sin(\beta x)] + [-\cosh(\beta L) + \cos(\beta L)][\cosh(\beta x) + \cos(\beta x)]\}$$

(G-43)

The first derivative is

$$\frac{dy}{dx} =$$

$$\hat{a}_1 \beta \{[\sinh(\beta L) + \sin(\beta L)][\cosh(\beta x) + \cos(\beta x)] + [-\cosh(\beta L) + \cos(\beta L)][\sinh(\beta x) - \sin(\beta x)]\}$$

(G-44)

The second derivative is

$$\frac{d^2 y}{dx^2} =$$

$$\hat{a}_1 \beta^2 \{[\sinh(\beta L) + \sin(\beta L)][\sinh(\beta x) - \sin(\beta x)] + [-\cosh(\beta L) + \cos(\beta L)][\cosh(\beta x) - \cos(\beta x)]\}$$

(G-45)